

On Liouville type theorems for the steady Navier-Stokes equations in \mathbb{R}^3

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Abstract

In this paper we prove three different Liouville type theorems for the steady Navier-Stokes equations in \mathbb{R}^3 . In the first theorem we improve logarithmically the well-known $L^{\frac{9}{2}}(\mathbb{R}^3)$ result. In the second theorem we present a sufficient condition for the triviality of the solution ($v = 0$) in terms of the head pressure, $Q = \frac{1}{2}|v|^2 + p$. The imposed integrability condition here has the same scaling property as the Dirichlet integral. In the last theorem we present Fubini type condition, which guarantee $v = 0$.

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1 Introduction

We consider the following stationary Navier-Stokes equations (NS) on \mathbb{R}^3 .

$$(NS) \begin{cases} (v \cdot \nabla)v = -\nabla p + \Delta v, \\ \operatorname{div} v = 0, \end{cases}$$

where $v(x) = (v_1(x), v_2(x), v_3(x))$ and $p = p(x)$ for all $x \in \mathbb{R}^3$. The system is equipped with the boundary condition:

$$|v(x)| \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty. \quad (1.1)$$

In addition to (1.1) one usually also assume following finiteness of the Dirichlet integral.

$$\int_{\mathbb{R}^3} |\nabla v|^2 dx < +\infty. \quad (1.2)$$

A long standing open question is if any weak solution of (NS) satisfying the conditions (1.1) and (1.2) is trivial (namely, $v = 0$ on \mathbb{R}^3). We refer the book by Galdi([2]) for the details on the motivations and historical backgrounds on the problem and the related results. As a partial progress to the problem we mention that the condition $v \in L^{\frac{9}{2}}(\mathbb{R}^3)$ implies that $v = 0$ (see Theorem X.9.5, pp.729 [2]). As shown in [1], a different condition $\Delta v \in L^{\frac{6}{5}}(\mathbb{R}^3)$ also imply $v = 0$. Another interesting progress, which shows that a solution $v \in BMO^{-1}(\mathbb{R}^3)$ to (NS), satisfying (1.2) is trivial is obtained very recently by Seregin in [6]. For the case of plane flows the problem is solved by Gilbarg and Weinberger in [3], while the special case of the axially symmetric 3D flows without swirl is studied recently by Korobkov, M. Pileckas and R. Russo in [5](see also [4]). In this paper we present three theorems, which present sufficient conditions to guarantee the triviality of the solution to (NS).

In the first theorem below we improve the above mentioned $L^{\frac{9}{2}}$ -result logarithmically.

Theorem 1.1. *Let $v \in L^1_{loc}(\mathbb{R}^3)$ be a distributional solution to (NS) such that*

$$\int_{\mathbb{R}^3} |v|^{\frac{9}{2}} \left\{ \log \left(2 + \frac{1}{|v|} \right) \right\}^{-1} dx < +\infty. \quad (1.3)$$

Then $v \equiv 0$.

For discussion of the next theorem we introduce the head pressure,

$$Q = \frac{1}{2}|v|^2 + p,$$

which has an important role in the study of the stationary Euler equations via the Bernoulli theorem. It is known(see e.g. Theorem X.5.1, pp. 688 [2]) that under the condition (1.1)-(1.2) we have $p(x) \rightarrow p_0$ as $|x| \rightarrow +\infty$, where p_0 is a constant, which implies that

$$Q(x) \rightarrow 0 \quad \text{as} \quad |x| \rightarrow +\infty \quad (1.4)$$

after re-defining $Q - p_0$ as the new head pressure. Our second theorem below assumes integrability of Q to conclude the triviality of v .

Theorem 1.2. *Let (v, p) be a smooth solution to (NS) satisfying (1.4). Let us set $M := \sup_{x \in \mathbb{R}^3} |Q(x)|$. Then, we have the following inequality.*

$$\int_{\mathbb{R}^3} \frac{|\nabla Q|^2}{|Q|} \left(\log \frac{eM}{|Q|} \right)^{-\alpha-1} dx \leq \frac{1}{\alpha} \int_{\mathbb{R}^3} |\omega|^2 dx \quad \forall \alpha > 0. \quad (1.5)$$

Moreover, suppose there holds the boundary conditions (1.1), (1.4) and

$$\int_{\mathbb{R}^3} \frac{|\nabla Q|^2}{|Q|} \left(\log \frac{eM}{|Q|} \right)^{-\alpha-1} dx = o \left(\frac{1}{\alpha} \right) \quad \text{as} \quad \alpha \rightarrow 0, \quad (1.6)$$

then $v = 0$ on \mathbb{R}^3 .

Remark 1.1. Since $|\nabla\sqrt{|Q|}|^2 = \frac{1}{4}\frac{|\nabla Q|^2}{|Q|}$, and $\sqrt{|Q|}$ has the same scaling as the velocity the integral $\int_{\mathbb{R}^3} \frac{|\nabla Q|^2}{|Q|} dx$ has the same scaling property as the Dirichlet integral in (1.2).

Our third result concerns on the Fubini type condition for suitable function $\Phi(x, y)$ for $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ to guarantee the triviality of the solution to (NS).

Theorem 1.3. *Let v be a smooth solution to (NS) on \mathbb{R}^3 satisfying (1.1) and set $\omega = \text{curl } v$. Suppose there exists $q \in [\frac{3}{2}, 3)$ such that $x \in L^q(\mathbb{R}^3)$. We set*

$$\Phi(x, y) := \frac{1}{4\pi} \frac{\omega(x) \cdot (x - y) \times (v(y) \times \omega(y))}{|x - y|^3} \quad (1.7)$$

for all $(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3$ with $x \neq y$. Then, it holds

$$\int_{\mathbb{R}^3} |\Phi(x, y)| dy + \int_{\mathbb{R}^3} |\Phi(x, y)| dx < \infty \quad \forall (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3. \quad (1.8)$$

Furthermore, if there holds

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(x, y) dx dy = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(x, y) dy dx, \quad (1.9)$$

then, $v = 0$ on \mathbb{R}^3 .

Remark 1.2. One can show that if $\omega \in L^{\frac{9}{5}}(\mathbb{R}^3)$ is satisfied together with (1.1), then (1.9) holds, and therefore v is trivial. Although this result follows immediately by applying the $L^{\frac{9}{2}}$ -result together with Sobolev inequality and the Calderon-Zygmund inequality, $\|v\|_{L^{\frac{9}{2}}} \leq C\|\nabla v\|_{L^{\frac{9}{5}}} \leq C\|\omega\|_{L^{\frac{9}{5}}}$. The above theorem provides us with different proof of this. In order to check this result we first recall the estimate of the Riesz potential on \mathbb{R}^3 ([8]),

$$\|I_\alpha(f)\|_{L^q} \leq C\|f\|_{L^p}, \quad \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{3}, \quad 1 \leq p < q < +\infty, \quad (1.10)$$

where

$$I_\alpha(f) := C \int_{\mathbb{R}^3} \frac{f(y)}{|x - y|^{3-\alpha}} dy, \quad 0 < \alpha < 3$$

for a positive constant $C = C(\alpha)$. Applying (1.10) with $\alpha = 1$, we obtain by the

Hölder inequality,

$$\begin{aligned}
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\Phi(x, y)| dy dx &\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|\omega(x)| |\omega(y)| |v(y)|}{|x - y|^2} dy dx \\
&\leq \left(\int_{\mathbb{R}^3} |\omega(x)|^{\frac{9}{5}} dx \right)^{\frac{5}{9}} \left\{ \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \frac{|\omega(y)| |v(y)|}{|x - y|^2} dy \right)^{\frac{9}{4}} dx \right\}^{\frac{4}{9}} \\
&\leq C \|\omega\|_{L^{\frac{9}{5}}} \left(\int_{\mathbb{R}^3} |\omega|^{\frac{9}{7}} |v|^{\frac{9}{7}} dx \right)^{\frac{7}{9}} \\
&\leq C \|\omega\|_{L^{\frac{9}{5}}} \left(\int_{\mathbb{R}^3} |\omega|^{\frac{9}{5}} dx \right)^{\frac{5}{9}} \left(\int_{\mathbb{R}^3} |v|^{\frac{9}{2}} dx \right)^{\frac{2}{9}} \\
&\leq C \|\omega\|_{L^{\frac{9}{5}}}^2 \|\nabla v\|_{L^{\frac{9}{5}}} \leq C \|\omega\|_{L^{\frac{9}{5}}}^3 < +\infty,
\end{aligned}$$

Thus, by the Fubini-Tonelli theorem, (1.9) holds.

2 Proof of the main theorems

Below we use the notation $A \lesssim B$ if there exists an absolute constant κ such that $A \leq \kappa B$.

2.1 Proof of Theorem 1.1

Definition 2.1. Let $\phi \in C^2(\mathbb{R})$ be an N-function, i. e. ϕ is an even function such that $\lim_{\tau \rightarrow 0} \phi'(\tau) = 0$, and $\lim_{\tau \rightarrow \infty} \phi'(\tau) = +\infty$. We say ϕ belongs to the class $N(p_0, p_1)$ ($1 < p_0 \leq p_1 < +\infty$) if for all $\tau \geq 0$

$$(p_0 - 1)\phi'(\tau) \leq \tau\phi''(\tau) \leq (p_1 - 1)\phi'(\tau). \quad (2.1)$$

Remark 2.2. It is well known that $\phi \in N(p_0, p_1)$ implies for all $\tau \geq 0$

$$\phi(\tau) \leq \tau\phi'(\tau) \leq p_1\phi(\tau). \quad (2.2)$$

We now define for $q > 1$

$$\phi_q(\tau) = \int_0^\tau \frac{\xi^{q-1}}{\log \frac{1+2\xi}{\xi}} d\xi, \quad \tau \geq 0.$$

We easily calculate,

$$\begin{aligned}
\phi_q'(\tau) &= \frac{\tau^{q-1}}{\log \frac{1+2\tau}{\tau}}, \\
\phi_q''(\tau) &= (q-1) \frac{\tau^{q-2}}{\log \frac{1+2\tau}{\tau}} + \frac{\tau^{q-2}}{\log^2 \frac{1+2\tau}{\tau}} \frac{1}{1+2\tau} \\
&= \frac{\phi_q'(\tau)}{\tau} \left((q-1) + \frac{1}{(1+2\tau) \log \frac{1+2\tau}{\tau}} \right).
\end{aligned}$$

Observing that $\frac{1}{(1+2\tau)\log\frac{1+2\tau}{\tau}} \leq \frac{1}{\log 2}$, we get for all $\tau \geq 0$

$$(q-1)\phi'_q(\tau) \leq \tau\phi''_q(\tau) \leq (q+(\log 2)^{-1}-1)\phi'_q(\tau). \quad (2.3)$$

This shows that $\phi \in N(q, q+(\log 2)^{-1})$, and according to (2.2) it holds

$$\phi_q(\tau) \sim \tau\phi'_q(\tau) = \frac{\tau^q}{\log\frac{2+\tau}{\tau}}. \quad (2.4)$$

Thus, (2.21) is equivalent to

$$\int_{\mathbb{R}^3} \phi_{\frac{q}{2}}(|v|)dx < +\infty. \quad (2.5)$$

Lemma 2.1. *For any constant $a > \frac{1}{2}$ we have*

$$\log a \frac{1+2\tau}{\tau} \sim \log \frac{1+2\tau}{\tau}. \quad (2.6)$$

Proof In case $a \geq 1$ we immediately get $\log a \frac{1+2\tau}{\tau} \geq \log \frac{1+2\tau}{\tau}$. For the reverse we get for all $0 < \tau \leq 1$,

$$\log a \frac{1+2\tau}{\tau} \leq \left(\frac{\log a}{\log 3} + 1 \right) \log \frac{1+2\tau}{\tau},$$

and for all $\tau > 1$

$$\log a \frac{1+2\tau}{\tau} \leq \log a + \log 3 \leq \frac{\log a + \log 3}{\log 2} \log \frac{1+2\tau}{\tau},$$

which proves the claim.

In case $a < 1$ we see that $\log a \frac{1+2\tau}{\tau} \leq \log \frac{1+2\tau}{\tau}$. On the other hand, we may choose $\tau_0 > 0$, such that

$$\log \frac{1+2\tau_0}{\tau_0} = \frac{1}{2} \left(1 + \frac{\log 2}{\log a^{-1}} \right) \log a^{-1}.$$

Then for $\tau \leq \tau_0$ we obtain

$$\begin{aligned} \log a \frac{1+2\tau}{\tau} &= -\log a^{-1} + \log \frac{1+2\tau}{\tau} = -2 \left(1 + \frac{\log 2}{\log a^{-1}} \right)^{-1} \log \frac{2+\tau_0}{\tau_0} + \log \frac{1+2\tau}{\tau} \\ &\geq \left[1 - 2 \left(1 + \frac{\log 2}{\log a^{-1}} \right)^{-1} \right] \log \frac{1+2\tau}{\tau} \\ &= \frac{\log 2 - \log a^{-1}}{\log 2 + \log a^{-1}} \log \frac{1+2\tau}{\tau}. \end{aligned}$$

For $\tau > \tau_0$ we easily see that

$$\begin{aligned} \log a \frac{1+2\tau}{\tau} &\geq \log 2 - \log a^{-1} = \frac{\log 2 - \log a^{-1}}{\log \frac{1+2\tau_0}{\tau_0}} \log \frac{1+2\tau_0}{\tau_0} \\ &\geq \frac{\log 2 - \log a^{-1}}{\log \frac{1+2\tau_0}{\tau_0}} \log \frac{1+2\tau}{\tau} \\ &= 2 \frac{\log 2 - \log a^{-1}}{\log 2 + \log a^{-1}} \log \frac{1+2\tau}{\tau}. \end{aligned}$$

Whence, the claim. ■

Lemma 2.2. For all $k \in \mathbb{N}$

$$\log \frac{1+2\tau}{\tau} \sim \log \frac{1+2\tau^k}{\tau^k}, \quad (2.7)$$

where the hidden constants depend on q and k only.

Proof In fact having $1+2\tau^k \leq (1+2\tau)^k \leq 2^{k-1}(1+2^k\tau^k) \leq 2^{2k-2}(1+2\tau^k)$ along with Lemma 2.1, we obtain

$$\log \frac{1+2\tau^k}{\tau^k} \leq k \log \frac{1+2\tau}{\tau} \leq \log 2^{2k-2} \frac{1+2\tau^k}{\tau^k} \lesssim \log \frac{1+2\tau^k}{\tau^k}.$$

This proves the claim. \blacksquare

Lemma 2.3. Let $f \in L^1(\mathbb{R}^3)$. Then for every $\varepsilon > 0$, there exists $R > \varepsilon^{-1}$, such that

$$\int_{B_R \setminus B_{R/2}} |f| dx \leq \frac{\varepsilon}{\log R}. \quad (2.8)$$

Proof Assume the assertion of the lemma is not true. Then there exists $\varepsilon > 0$ such that for all $R \geq \varepsilon^{-1}$ (2.8) does not hold. This implies for all $k \geq N$ with $2^k \geq \varepsilon^{-1}$

$$\int_{B_{2^k} \setminus B_{2^{k-1}}} |f| dx \geq \frac{\varepsilon}{k \log 2}.$$

However the sum of right-hand side from $k = N$ to ∞ is infinite which clearly contradicts to $f \in L^1(\mathbb{R}^3)$. Thus, the assumption is not true and therefore the assertion of the lemma holds. \blacksquare

In view of (1.3) we easily see that $v \in L_{\text{loc}}^{\frac{9}{2}}(\mathbb{R}^3)$. By using a standard mollifying argument we verify that $v \in W_{\text{loc}}^{1,2}(\mathbb{R}^3)$, and therefore $v \in C^\infty(\mathbb{R}^3)$ and $p \in C^\infty(\mathbb{R}^3)$. In particular, we have for all $\zeta \in C_c^\infty(\mathbb{R}^3)$

$$\int_{\mathbb{R}^3} |\nabla v|^2 \zeta dx = \frac{1}{2} \int_{\mathbb{R}^3} |v|^2 \Delta \zeta dx + \frac{1}{2} \int_{\mathbb{R}^3} |v|^2 v \cdot \nabla \zeta dx + \int_{\mathbb{R}^3} p v \cdot \nabla \zeta dx. \quad (2.9)$$

On the basis of (2.9) we have the following Caccioppoli-type inequality.

$$\int_{B_R} |\nabla v|^2 dx \lesssim R^{-1} \left\{ 1 + \int_{B_{2R}} |v|^3 dx \right\}. \quad (2.10)$$

Proof of (2.10): Let $R \leq r < \rho \leq 2R$. Into (2.9) we insert a off function $\zeta \in C_c^\infty(B_\rho)$ such that $\zeta \equiv 1$ on B_r , $0 \leq \zeta \leq 1$ in \mathbb{R}^3 and $|\nabla \zeta|^2 + |\nabla^2 \zeta| \lesssim (\rho - r)^{-2}$. This together with Hölder's inequality and Young's inequality immediately gives

$$\begin{aligned} \int_{B_r} |\nabla v|^2 dx &\lesssim (\rho - r)^{-2} \int_{B_\rho} |v|^2 dx + (\rho - r)^{-1} \int_{B_\rho} |v|^3 dx + (\rho - r)^{-1} \int_{B_\rho} |p - p_{B_\rho}| |v| dx \\ &\lesssim (\rho - r)^{-1} \left\{ 1 + \int_{B_{2R}} |v|^3 dx \right\} + (\rho - r)^{-1} \int_{B_\rho} |p - p_{B_\rho}| |v| dx. \end{aligned} \quad (2.11)$$

Using Hölder's inequality, Young's inequality and consulting Theorem III.3.1, Theorem III.5.2 of [2], we estimate the last integral involving the pressure as follows

$$\begin{aligned}
& (\rho - r)^{-1} \int_{B_\rho} |p - p_{B_\rho}| |v| dx \\
& \lesssim (\rho - r)^{-1} \left(\int_{B_\rho} |\nabla v|^{\frac{2}{3}} dx + \int_{B_\rho} |v|^3 dx \right)^{\frac{2}{3}} \left(\int_{B_\rho} |v|^3 dx \right)^{\frac{1}{3}} \\
& \lesssim \rho^{1/2} (\rho - r)^{-1} \left(\int_{B_\rho} |\nabla v|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_{2R}} |v|^3 dx \right)^{\frac{1}{3}} + (\rho - r)^{-1} \int_{B_{2R}} |v|^3 dx \\
& \lesssim \delta \int_{B_\rho} |\nabla v|^2 dx + \rho (\rho - r)^{-2} \left(\int_{B_{2R}} |v|^3 dx \right)^{\frac{2}{3}} + (\rho - r)^{-1} \int_{B_{2R}} |v|^3 dx \\
& \lesssim \delta \int_{B_\rho} |\nabla v|^2 dx + (\rho - r)^{-1} \left\{ 1 + \int_{B_{2R}} |v|^3 dx \right\}.
\end{aligned}$$

Inserting this inequality into the right-hand side of (2.11), we arrive at

$$\int_{B_r} |\nabla v|^2 dx \lesssim (\rho - r)^{-1} \left\{ 1 + \int_{B_{2R}} |v|^3 dx \right\} + \delta \int_{B_\rho} |\nabla v|^2 dx. \quad (2.12)$$

In (2.12) taking $\delta > 0$ sufficiently small, and applying a well known iteration argument, we obtain (2.10). This completes the proof of (2.10). \blacksquare

Proof of Theorem 1.1 Let $\varepsilon > 0$ be arbitrarily chosen, but fixed. Thanks to Lemma 2.3, in view of (2.8) we may choose $R \geq \varepsilon^{-1}$ such that

$$\int_{B_R \setminus B_{R/2}} \phi_{\frac{9}{2}}(|v|) dx \leq \frac{\varepsilon}{\log R}. \quad (2.13)$$

Let $\zeta \in C_c^\infty(B_R)$ be a cut off function such that $0 \leq \zeta \leq 1$ in B_R , $\zeta \equiv 1$ on $B_{R/2}$, and $|\nabla \zeta| \lesssim R^{-1}$, $|\nabla^2 \zeta| \lesssim R^{-2}$. Then from (2.9) we deduce

$$\begin{aligned}
\int_{B_{R/2}} |\nabla v|^2 dx & \lesssim R^{-2} \int_{B_R \setminus B_{R/2}} |v|^2 dx + R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3 dx + R^{-1} \int_{B_R \setminus B_{R/2}} |p - p_{B_R \setminus B_{R/2}}| |v| dx \\
& \quad (2.14)
\end{aligned}$$

Using Hölder's inequality, Young's inequality and consulting [2], we estimate the last

integral involving the pressure as follows

$$\begin{aligned}
& R^{-1} \int_{B_R \setminus B_{R/2}} |p - p_{B_R \setminus B_{R/2}}| |v| dx \\
& \lesssim R^{-1} \left(\int_{B_R \setminus B_{R/2}} |\nabla v|^{\frac{3}{2}} dx + \int_{B_R \setminus B_{R/2}} |v|^3 dx \right)^{\frac{2}{3}} \left(\int_{B_R \setminus B_{R/2}} |v|^3 dx \right)^{\frac{1}{3}} \\
& \lesssim R^{-\frac{1}{2}} \left(\int_{B_R} |\nabla v|^2 dx \right)^{\frac{1}{2}} \left(\int_{B_R \setminus B_{R/2}} |v|^3 dx \right)^{\frac{1}{3}} + R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3 dx \\
& \lesssim R^{-\frac{1}{3}} \left(\int_{B_R} |\nabla v|^2 dx \right)^{\frac{3}{4}} + R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3 dx \\
& \lesssim R^{-\frac{1}{8}} + R^{-\frac{1}{6}} \int_{B_R} |\nabla v|^2 dx + R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3 dx.
\end{aligned}$$

Once more using Hölder's inequality along with Young's inequality we easily find

$$R^{-2} \int_{B_R \setminus B_{R/2}} |v|^2 dx \leq R^{-1} + R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3 dx.$$

Inserting the last two inequalities into the right-hand side of (2.14), we arrive at

$$\int_{B_{R/2}} |\nabla v|^2 dx \lesssim R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3 dx + R^{-\frac{1}{8}} + R^{-\frac{1}{6}} \int_{B_R} |\nabla v|^2 dx. \quad (2.15)$$

We now estimate the last integral on the right-hand side of (2.15) by means of (2.10). This implies

$$\int_{B_{R/2}} |\nabla v|^2 dx \lesssim R^{-1} \int_{B_R \setminus B_{R/2}} |v|^3 dx + R^{-\frac{1}{8}} + R^{-\frac{7}{6}} \int_{B_{2R}} |v|^3 dx. \quad (2.16)$$

By our assumption (1.3) we know that $v \in L^q(\mathbb{R}^3)$ for all $q > \frac{9}{2}$. This follows from standard regularity theory of the steady Navier-Stokes equations (e.g. see [7]). For $\frac{9}{2} < q < \frac{54}{11}$ we find with the help of Jensen's inequality

$$R^{-\frac{1}{8}} + R^{-\frac{7}{6}} \int_{B_{2R}} |v|^3 dx \lesssim R^{-\frac{1}{8}} + R^{\frac{3q-9}{q}-\frac{7}{6}} \|v\|_q \rightarrow 0 \quad \text{as } R \rightarrow +\infty. \quad (2.17)$$

Noting that $\phi_{3/2}$ is convex, applying Jensen's inequality, we get

$$\begin{aligned}\phi_{\frac{3}{2}}\left(\frac{8}{7R \operatorname{meas}(B_1)} \int_{B_R \setminus B_{R/2}} |v|^3 dx\right) &= \phi_{\frac{3}{2}}\left(R^2 \int_{B_R \setminus B_{R/2}} |v|^3 dx\right) \\ &\leq \int_{B_R \setminus B_{R/2}} \phi_{\frac{3}{2}}(R^2 |v|^3) dx \\ &\lesssim \int_{B_R \setminus B_{R/2}} \frac{|v|^{\frac{9}{2}}}{\log \frac{1+2R^2|v|^3}{R^2|v|^3}} dx.\end{aligned}$$

We split the integral on the right-hand side into two parts by setting

$$\begin{aligned}A_1 &= \{x \in B_R \setminus B_{R/2} \mid |v|^3 \leq \varepsilon R^{-2}\}, \\ A_2 &= \{x \in B_R \setminus B_{R/2} \mid |v|^3 > \varepsilon R^{-2}\}.\end{aligned}$$

Firstly, we easily see that

$$\int_{A_1} \frac{|v|^{\frac{9}{2}}}{\log \frac{1+2R^2|v|^3}{R^2|v|^3}} dx \lesssim \varepsilon^{\frac{3}{2}}.$$

Secondly, with help of Lemma 2.2 and recalling that $R \geq \frac{1}{\varepsilon}$ we have in A_2

$$\begin{aligned}4 \log R &\geq \log R^2 + \log \frac{1}{\varepsilon} + \log 2 = \log 2 \frac{R^2}{\varepsilon} \geq \log \frac{1+2\varepsilon R^{-2}}{\varepsilon R^{-2}} \\ &\geq \log \frac{1+2|v|^3}{|v|^3} \gtrsim \log \frac{1+2|v|}{|v|}.\end{aligned}$$

With this estimate along with (2.13) we get

$$\begin{aligned}\int_{A_2} \frac{|v|^{\frac{9}{2}}}{\log \frac{1+2R^2|v|^3}{R^2|v|^3}} dx &\lesssim \frac{1}{\log 2} \int_{A_2} |v|^{\frac{9}{2}} dx \lesssim \frac{\log R}{\log 2} \int_{B_R \setminus B_{R/2}} \frac{|v|^{\frac{9}{2}}}{\log \frac{1+2|v|}{|v|}} dx \\ &\lesssim \log R \int_{B_R \setminus B_{R/2}} \phi_{\frac{9}{2}}(|v|) dx \lesssim \varepsilon.\end{aligned}$$

Accordingly,

$$\phi_{\frac{3}{2}}\left(\frac{8}{7R \operatorname{meas}(B_1)} \int_{B_R \setminus B_{R/2}} |v|^3 dx\right) \lesssim \varepsilon.$$

Thus, in view (2.16) together with the estimates we have just obtained we are able to chose a sequence $R_k \rightarrow +\infty$ as $k \rightarrow +\infty$, such that

$$\int_{B_{R_k/2}} |\nabla v|^2 dx \rightarrow 0 \quad \text{as } k \rightarrow +\infty,$$

which yields $\nabla v = 0$ and therefore $v \equiv \text{const} = 0$. ■

2.2 Proof of Theorem 1.2

Proof of Theorem 1.2 Let us denote the vorticity $\omega = \text{curl } v$. Then, it is well-known that from (NS) that the following equation holds true.

$$\Delta Q - v \cdot \nabla Q = |\omega|^2. \quad (2.18)$$

Under the condition (1.4) we have $Q(x) \leq 0$ for all $x \in \mathbb{R}^3$ by the maximum principle applied to (2.18). Moreover, by the maximum principle again, either $Q(x) \equiv 0$ on \mathbb{R}^3 , or $Q(x) < 0$ for all $x \in \mathbb{R}^3$. Indeed, any point $x_0 \in \mathbb{R}^3$ such that $Q(x_0) = 0$ is a point of local maximum, which is not allowed unless $Q \equiv 0$ by the maximum principle. Let $Q(x) \not\equiv 0$ on \mathbb{R}^3 , then without the loss of generality we may assume $|Q(x)| > 0$ for all $x \in \mathbb{R}^3$. We set $\sup_{x \in \mathbb{R}^3} |Q| = M > 0$. Let $f \in C(\mathbb{R})$. For $\lambda \in [0, M)$ we set $D_\lambda = \{x \in \mathbb{R}^3 \mid |Q(x)| > \lambda\}$. Then, we compute

$$\begin{aligned} \int_{D_\lambda} f(Q(x)) v \cdot \nabla Q \, dx &= \int_{D_\lambda} v \cdot \nabla \left(\int_0^{Q(x)} f(q) \, dq \right) \, dx \\ &= \int_{D_\lambda} \text{div} \left(v \int_0^{Q(x)} f(q) \, dq \right) \, dx = \int_{\partial D_\lambda} \left(\int_0^{Q(x)} f(q) \, dq \right) v \cdot \nu \, dS \\ &= \int_0^\lambda f(q) \, dq \int_{\partial D_\lambda} v \cdot \nu \, dS = \int_0^\lambda f(q) \, dq \int_{D_\lambda} \text{div } v \, dx = 0. \end{aligned} \quad (2.19)$$

where $\nu = \nabla Q / |\nabla Q|$ is the outward unit normal vector on ∂D_λ . For $\lambda \in (0, M)$ we integrate (2.18) over D_λ . Then, using the fact (2.19), we have

$$\begin{aligned} \int_{D_\lambda} |\omega|^2 \, dx &= \int_{D_\lambda} \Delta Q \, dx = \int_{\partial D_\lambda} \frac{\partial Q}{\partial \nu} \, dS \\ &= \int_{\partial D_\lambda} |\nabla Q| \, dS. \end{aligned} \quad (2.20)$$

Using the co-area formula, we obtain

$$\begin{aligned} \int_{D_\lambda} \frac{|\nabla Q|^2}{|Q|} \left(\log \frac{eM}{|Q|} \right)^{-\alpha-1} \, dx &= \int_\lambda^M \int_{\partial D_q} \frac{|\nabla Q|}{|Q|} \left(\log \frac{eM}{|Q|} \right)^{-\alpha-1} \, dS \, dq \\ &= \int_\lambda^M \frac{1}{q} \left(\log \frac{eM}{q} \right)^{-\alpha-1} \int_{\partial D_q} |\nabla Q| \, dS \, dq \\ &\leq \int_\lambda^M \frac{1}{q} \left(\log \frac{eM}{q} \right)^{-\alpha-1} \, dq \int_{\partial D_\lambda} |\nabla Q| \, dS \\ &= \frac{1}{\alpha} \left\{ 1 - \left(\log \frac{eM}{\lambda} \right)^{-\alpha} \right\} \int_{D_\lambda} |\omega|^2 \, dx \\ &\leq \frac{1}{\alpha} \int_{\mathbb{R}^3} |\omega|^2 \, dx, \end{aligned}$$

where we used (2.20) in the fourth line. Passing $\lambda \rightarrow 0$, and applying the monotone convergence theorem, we obtain (1.5). Next, we assume (1.6) holds. We consider a

standard cut-off function $\sigma \in C_0^\infty([0, \infty))$ such that $\sigma(s) = 1$ if $s < 1$, and $\sigma(s) = 0$ if $s > 2$, and $0 \leq \sigma(s) \leq 1$ for $1 < s < 2$. For each $\alpha \in (0, 1)$ we define $\sigma_\alpha(x) := \sigma_\alpha(Q(x)) \in C_0^\infty(\mathbb{R}^3)$ by

$$\sigma_\alpha(x) = 1 - \sigma \left\{ 3 \left(\log \frac{eM}{|Q(x)|} \right)^{-\alpha} \right\}.$$

We note that

$$\begin{cases} \sigma_\alpha(x) = 1, & \text{if } |Q(x)| \geq Me^{1-(\frac{3}{2})^{\frac{1}{\alpha}}}, \\ 0 < \sigma_\alpha(x) < 1, & \text{if } Me^{1-3^{\frac{1}{\alpha}}} < |Q(x)| < Me^{1-(\frac{3}{2})^{\frac{1}{\alpha}}}, \\ \sigma_\alpha(x) = 0, & \text{if } |Q(x)| \leq Me^{1-3^{\frac{1}{\alpha}}}. \end{cases}$$

We multiply (2.18) by σ_α , and integrate it over \mathbb{R}^3 . Then, the convection term vanishes by (2.19). Let $\alpha_1 > 0$ be fixed. For all $\alpha > \alpha_1$ we have

$$\begin{aligned} \int_{\mathbb{R}^3} |\omega|^2 \sigma_{\alpha_1}(x) dx &\leq \int_{\mathbb{R}^3} |\omega|^2 \sigma_\alpha(x) dx = \int_{\mathbb{R}^3} \Delta Q \sigma_\alpha(x) dx \\ &= -3\alpha \int_{\{Me^{1-3^{\frac{1}{\alpha}}} < |Q(x)| < Me^{1-(\frac{3}{2})^{\frac{1}{\alpha}}}\}} \frac{|\nabla Q|^2}{|Q|} \left(\log \frac{eM}{|Q|} \right)^{-\alpha-1} \sigma' \left\{ 3 \left(\log \frac{eM}{|Q(x)|} \right)^{-\alpha} \right\} dx \\ &\leq 3\alpha \sup_{1 \leq s \leq 2} |\sigma'(s)| \int_{\{Me^{1-3^{\frac{1}{\alpha}}} < |Q(x)| < Me^{1-(\frac{3}{2})^{\frac{1}{\alpha}}}\}} \frac{|\nabla Q|^2}{|Q|} \left(\log \frac{eM}{|Q|} \right)^{-\alpha-1} dx \\ &\rightarrow 0 \quad \text{as } \alpha \rightarrow 0. \end{aligned}$$

Hence, we have shown $\int_{\mathbb{R}^3} |\omega|^2 \sigma_{\alpha_1}(x) dx = 0$ for all $\alpha_1 > 0$, which implies that $\omega = 0$ on \mathbb{R}^3 . This, combined with the fact $\operatorname{div} v = 0$ implies that v is a harmonic function on \mathbb{R}^3 . The boundary condition, together with the Liouville theorem for harmonic function, leads us to conclude $v = 0$ on \mathbb{R}^3 . \blacksquare

2.3 Proof of Theorem 1.3

We first establish integrability conditions on the vector fields for the Biot-Savart's formula in \mathbb{R}^3 .

Proposition 2.1. *Let $\xi = \xi(x) = (\xi_1(x), \xi_2(x), \xi_3(x))$ and $\eta = \eta(x) = (\eta_1(x), \eta_2(x), \eta_3(x))$ be smooth vector fields on \mathbb{R}^3 . Suppose there exists $q \in [1, 3)$ such that $\eta \in L^q(\mathbb{R}^3)$. Let ξ solve*

$$\Delta \xi = -\nabla \times \eta, \tag{2.21}$$

under the boundary condition; either

$$|\xi(x)| \rightarrow 0 \quad \text{as } |x| \rightarrow +\infty, \tag{2.22}$$

or

$$\xi \in L^s(\mathbb{R}^3) \quad \text{for some } s \in [1, \infty). \quad (2.23)$$

Then, the solution of (2.21) is given by

$$\xi(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \eta(y)}{|x-y|^3} dy \quad \forall x \in \mathbb{R}^3. \quad (2.24)$$

Proof Let $\sigma \in C_0^\infty(\mathbb{R}^3)$ be the cut-off function defined in the proof of Theorem 1.1. For each $R > 0$ we define $\sigma_R(x) := \sigma\left(\frac{|x|}{R}\right)$. Given $\epsilon > 0$ we denote $B_\epsilon(y) = \{x \in \mathbb{R}^3 \mid |x-y| < \epsilon\}$. Let us fix $y \in \mathbb{R}^3$ and $\epsilon \in (0, \frac{R}{2})$. We multiply (2.21) by $\frac{\sigma_R(|x-y|)}{|x-y|}$, and integrate it with respect to the variable x over $\mathbb{R}^3 \setminus B_\epsilon(y)$. Then,

$$\int_{\{|x-y|>\epsilon\}} \frac{\Delta \xi \sigma_R}{|x-y|} dx = - \int_{\{|x-y|>\epsilon\}} \frac{\sigma_R \nabla \times \eta(y)}{|x-y|} dx. \quad (2.25)$$

Since $\Delta \frac{1}{|x-y|} = 0$ on $\mathbb{R}^3 \setminus B_\epsilon(y)$, one has

$$\begin{aligned} \frac{\Delta \xi \sigma_R}{|x-y|} &= \sum_{i=0}^3 \partial_{x_i} \left(\frac{\partial_{x_i} \xi \sigma_R}{|x-y|} \right) - \sum_{i=0}^3 \partial_{x_i} \left(\frac{\xi \partial_{x_i} \sigma_R}{|x-y|} \right) \\ &\quad - \sum_{i=0}^3 \partial_{x_i} \left(\xi \sigma_R \partial_{x_i} \left(\frac{1}{|x-y|} \right) \right) + \frac{\xi \Delta \sigma_R}{|x-y|} + 2 \sum_{i=0}^3 \xi \partial_{x_i} \left(\frac{1}{|x-y|} \right) \partial_{x_i} \sigma_R. \end{aligned}$$

Therefore, applying the divergence theorem, and observing $\partial_\nu \sigma_R = 0$ on $\partial B_\epsilon(y)$, we have

$$\begin{aligned} \int_{\{|x-y|>\epsilon\}} \frac{\Delta \xi \sigma_R}{|x-y|} dx &= \int_{\{|x-y|=\epsilon\}} \frac{\partial_\nu \xi}{|x-y|} dS \\ &\quad - \int_{\{|x-y|=\epsilon\}} \frac{\xi}{|x-y|^2} dS + \int_{\{|x-y|>\epsilon\}} \frac{\xi \Delta \sigma_R}{|x-y|} dS \\ &\quad - 2 \int_{\{|x-y|>\epsilon\}} \frac{(x-y) \cdot \nabla \sigma_R \xi}{|x-y|^3} dS \end{aligned} \quad (2.26)$$

where $\partial_\nu(\cdot)$ denotes the outward normal derivative on $\partial B_\epsilon(y)$. Passing $\epsilon \rightarrow 0$, one can easily compute that

$$\begin{aligned} \text{RHS of (2.26)} &\rightarrow -4\pi \xi(y) + \int_{\mathbb{R}^3} \frac{\xi \Delta \sigma_R}{|x-y|} dx - 2 \int_{\mathbb{R}^3} \frac{(x-y) \cdot \nabla \sigma_R \xi}{|x-y|^3} dx \\ &:= I_1 + I_2 + I_3 \end{aligned} \quad (2.27)$$

Next, using the formula

$$\frac{\sigma_R \nabla \times \eta}{|x-y|} = \nabla \times \left(\frac{\sigma_R \eta}{|x-y|} \right) - \frac{\nabla \sigma_R \times \eta}{|x-y|} + \frac{(x-y) \times \eta \sigma_R}{|x-y|^3},$$

and using the divergence theorem, we obtain the following representation for the right hand side of (2.25).

$$\begin{aligned} \int_{\{|x-y|>\epsilon\}} \frac{\sigma_R \nabla \times \eta}{|x-y|} dx &= \int_{\{|x-y|=\epsilon\}} \nu \times \left(\frac{\eta}{|x-y|} \right) dS \\ &\quad - \int_{\{|x-y|>\epsilon\}} \frac{\nabla \sigma_R \times \eta}{|x-y|} dx + \int_{\{|x-y|>\epsilon\}} \frac{(x-y) \times \eta \sigma_R}{|x-y|^3} dx, \end{aligned} \quad (2.28)$$

where we denoted $\nu = \frac{y-x}{|y-x|}$, the outward unit normal vector on $\partial B_\epsilon(y)$. Passing $\epsilon \rightarrow 0$, we easily deduce

$$\begin{aligned} \text{RHS of (2.28)} &\rightarrow - \int_{\mathbb{R}^3} \frac{\nabla \sigma_R \times \eta}{|x-y|} dx - 2 \int_{\mathbb{R}^3} \frac{(x-y) \times \eta \sigma_R}{|x-y|^3} dx \\ &:= J_1 + J_2 \quad \text{as } \epsilon \rightarrow 0. \end{aligned} \quad (2.29)$$

We now pass $R \rightarrow \infty$ for each term of (2.27) and (2.29) respectively below. Under the boundary condition (2.22) we estimate:

$$\begin{aligned} |I_2| &\leq \int_{\{R \leq |x-y| \leq 2R\}} \frac{|\xi(x)| |\Delta \sigma_R(x-y)|}{|x-y|} dx \\ &\leq \frac{\|\Delta \sigma\|_{L^\infty}}{R^2} \sup_{R \leq |x-y| \leq 2R} |\xi(x)| \left(\int_{\{R \leq |x-y| \leq 2R\}} dx \right)^{\frac{2}{3}} \left(\int_{\{R \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^3} \right)^{\frac{1}{3}} \\ &\lesssim \|\Delta \sigma\|_{L^\infty} \left(\int_R^{2R} \frac{dr}{r} \right)^{\frac{1}{3}} \sup_{R \leq |x-y| \leq 2R} |\xi(x)| \rightarrow 0 \end{aligned}$$

as $R \rightarrow \infty$ by the assumption (2.22), while under the condition (2.23) we have

$$\begin{aligned} |I_2| &\leq \int_{\{R \leq |x-y| \leq 2R\}} \frac{|\xi(x)| |\Delta \sigma_R(x-y)|}{|x-y|} dx \\ &\leq \frac{\|\Delta \sigma\|_{L^\infty}}{R^2} \|\xi\|_{L^s} \left(\int_{\{R \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^{\frac{s}{s-1}}} \right)^{\frac{s-1}{s}} \\ &\lesssim R^{-\frac{3}{s}} \|\Delta \sigma\|_{L^\infty} \|\xi\|_{L^s} \rightarrow 0 \end{aligned}$$

as $R \rightarrow \infty$. Similarly, under (2.22)

$$\begin{aligned} |I_3| &\leq 2 \int_{\{R \leq |x-y| \leq 2R\}} \frac{|\xi(x)| |\nabla \sigma_R(x-y)|}{|x-y|^2} dx \\ &\lesssim \frac{\|\nabla \sigma\|_{L^\infty}}{R} \sup_{R \leq |x-y| \leq 2R} |\xi(x)| \left(\int_{\{R \leq |x-y| \leq 2R\}} dx \right)^{\frac{1}{3}} \left(\int_{\{R \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^3} \right)^{\frac{2}{3}} \\ &\lesssim \|\nabla \sigma\|_{L^\infty} \left(\int_R^{2R} \frac{dr}{r} \right)^{\frac{2}{3}} \sup_{R \leq |x-y| \leq 2R} |\xi(x)| \rightarrow 0 \end{aligned}$$

as $R \rightarrow \infty$, while under the condition (2.23) we estimate

$$\begin{aligned}
|I_3| &\leq 2 \int_{\{R \leq |x-y| \leq 2R\}} \frac{|\xi(x)| |\nabla \sigma_R(x-y)|}{|x-y|^2} dx \\
&\lesssim \frac{\|\nabla \sigma\|_{L^\infty}}{R^2} \|\xi\|_{L^s} \left(\int_{\{R \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^{\frac{2s}{s-1}}} \right)^{\frac{s-1}{s}} \\
&\lesssim R^{-\frac{3}{s}} \|\nabla \sigma\|_{L^\infty} \|\xi\|_{L^s} \rightarrow 0
\end{aligned}$$

as $R \rightarrow \infty$. Therefore, the right hand side of (2.26) converges to $-4\pi\xi(y)$ as $R \rightarrow \infty$. For J_1, J_2 we estimate

$$\begin{aligned}
|J_1| &\leq \int_{\{R \leq |x-y| \leq 2R\}} \frac{|\nabla \sigma_R| |\eta|}{|x-y|} dx \\
&\leq \frac{\|\nabla \sigma\|_{L^\infty}}{R} \|\eta\|_{L^q(R \leq |x-y| \leq 2R)} \left(\int_{\{R \leq |x-y| \leq 2R\}} \frac{dx}{|x-y|^{\frac{q}{q-1}}} \right)^{\frac{q-1}{q}} \\
&\lesssim \|\nabla \sigma\|_{L^\infty} \|\eta\|_{L^q(R \leq |x-y| \leq 2R)} R^{-\frac{2}{q}} \rightarrow 0
\end{aligned}$$

as $R \rightarrow \infty$. In passing $R \rightarrow \infty$ in J_2 of (2.29), in order to use the dominated convergence theorem, we estimate

$$\begin{aligned}
\int_{\mathbb{R}^3} \left| \frac{(x-y) \times \eta(y)}{|x-y|^3} \right| dx &\leq \int_{\{|x-y| < 1\}} \frac{|\eta|}{|x-y|^2} dx + \int_{\{|x-y| \geq 1\}} \frac{|\eta|}{|x-y|^2} dx \\
&:= J_{21} + J_{22}.
\end{aligned} \tag{2.30}$$

J_{21} is easy to handle as follows.

$$J_{21} \leq \|\eta\|_{L^\infty(B_1(y))} \int_{\{|x-y| < 1\}} \frac{dx}{|x-y|^2} = 4\pi \|\eta\|_{L^\infty(B_1(y))} < +\infty \tag{2.31}$$

For J_{22} we estimate

$$\begin{aligned}
J_{22} &\leq \left(\int_{\mathbb{R}^3} |\eta|^q \right)^{\frac{1}{q}} \left(\int_{\{|x-y| > 1\}} \frac{dx}{|x-y|^{\frac{2q}{q-1}}} \right)^{\frac{q-1}{q}} \\
&\lesssim \|\eta\|_{L^q} \left(\int_1^\infty r^{\frac{-2}{q-1}} dr \right)^{\frac{q-1}{q}} < +\infty,
\end{aligned} \tag{2.32}$$

if $1 < q < 3$. In the case of $q = 1$ we estimate simply

$$J_{22} \leq \int_{\{|x-y| > 1\}} |\eta| dx \leq \|\eta\|_{L^1}. \tag{2.33}$$

Estimates of (2.30)-(2.33) imply

$$\int_{\mathbb{R}^3} \left| \frac{(x-y) \times \eta(y)}{|x-y|^3} \right| dx < +\infty.$$

Summarising the above computations, one can pass first $\epsilon \rightarrow 0$, and then $R \rightarrow +\infty$ in (2.25), applying the dominated convergence theorem, to obtain finally (2.24). ■

Corollary 2.1. *Let v be a smooth solution to (NS) satisfying (1.1) such that $\omega \in L^q(\mathbb{R}^3)$ for some $q \in [\frac{3}{2}, 3)$. Then, we have*

$$v(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times \omega(y)}{|x-y|^3} dy, \quad (2.34)$$

and

$$\omega(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times (v(y) \times \omega(y))}{|x-y|^3} dy. \quad (2.35)$$

Proof Taking curl of the defining equation of the vorticity, $\nabla \times v = \omega$, using $\operatorname{div} v = 0$, we have

$$\Delta v = -\nabla \times \omega,$$

which provides us with (2.34) immediately by application of Proposition 2.1. In order to show (2.35) we recall that, using the vector identity $\frac{1}{2}\nabla|v|^2 = (v \cdot \nabla)v + v \times (\nabla \times v)$, one can rewrite (NS) as

$$-v \times \omega = -\nabla \left(p + \frac{1}{2}|v|^2 \right) + \Delta v.$$

Taking curl on this, we obtain

$$\Delta \omega = -\nabla \times (v \times \omega).$$

The formula (2.35) is deduced immediately from this equations by applying the proposition 2.1. For the allowed range of q we recall the Sobolev and the Calderon-Zygmund inequalities([8]),

$$\|v\|_{L^{\frac{3q}{3-q}}} \lesssim \|\nabla v\|_{L^q} \lesssim \|\omega\|_{L^q}, \quad 1 < q < 3, \quad (2.36)$$

which imply $v \times \omega \in L^{\frac{3q}{6-q}}(\mathbb{R}^3)$ if $\omega \in L^q(\mathbb{R}^3)$. We also note that $\frac{3}{2} \leq q < 3$ if and only if $1 \leq \frac{3q}{6-q} < 3$. ■

Proof of Theorem 1.3 Under the hypothesis (1.1) and $\omega \in L^q(\mathbb{R}^3)$ with $q \in [\frac{3}{2}, 3)$ both of the relations (2.34) and (2.35) are valid. We first prove the following.

Claim: For each $x, y \in \mathbb{R}^3$

$$0 \leq |\omega(x)|^2 = \int_{\mathbb{R}^3} \Phi(x, y) dy \leq \int_{\mathbb{R}^3} |\Phi(x, y)| dy < +\infty, \quad (2.37)$$

and

$$0 = \int_{\mathbb{R}^3} \Phi(x, y) dx \leq \int_{\mathbb{R}^3} |\Phi(x, y)| dx < +\infty. \quad (2.38)$$

Proof of (1.8): Decomposing the integral and using the Hölder inequality, we estimate

$$\begin{aligned}
\int_{\mathbb{R}^3} |\Phi(x, y)| dy &\leq |\omega(x)| \left(\int_{\{|x-y| \leq 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy + \int_{\{|x-y| > 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy \right) \\
&\leq |\omega(x)| \|v\|_{L^\infty(B_1(x))} \|\omega\|_{L^\infty(B_1(x))} \int_{\{|x-y| \leq 1\}} \frac{dy}{|x-y|^2} \\
&\quad + |\omega(x)| \|v\|_{L^{\frac{3q}{3-q}}} \|\omega\|_{L^q} \left(\int_{\{|x-y| \geq 1\}} \frac{dy}{|x-y|^{\frac{6q}{4q-6}}} \right)^{\frac{4q-6}{3q}} \\
&\lesssim |\omega(x)| \|v\|_{L^\infty(B_1(x))} \|\omega\|_{L^\infty(B_1(x))} \\
&\quad + |\omega(x)| \|\omega\|_{L^q}^2 \left(\int_1^\infty r^{\frac{q-6}{2q-3}} dr \right)^{\frac{4q-6}{3q}} < +\infty, \tag{2.39}
\end{aligned}$$

where we used (2.36) and the fact that $\frac{q-6}{3q-3} < -1$ if $\frac{3}{2} < q < 3$. In the case $q = \frac{3}{2}$ we estimate, instead,

$$\begin{aligned}
\int_{\mathbb{R}^3} |\Phi(x, y)| dy &\leq |\omega(x)| \left(\int_{\{|x-y| \leq 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy + \int_{\{|x-y| > 1\}} \frac{|v(y)||\omega(y)|}{|x-y|^2} dy \right) \\
&\leq |\omega(x)| \|v\|_{L^\infty(B_1(x))} \|\omega\|_{L^\infty(B_1(x))} + |\omega(x)| \|v\|_{L^3} \|\omega\|_{L^{\frac{3}{2}}} < +\infty. \tag{2.40}
\end{aligned}$$

We also have

$$\begin{aligned}
\int_{\mathbb{R}^3} |\Phi(x, y)| dx &\leq |v(y)||\omega(y)| \left(\int_{\{|x-y| \leq 1\}} \frac{|\omega(x)|}{|x-y|^2} dx + \int_{\{|x-y| > 1\}} \frac{|\omega(x)|}{|x-y|^2} dx \right) \\
&\lesssim |v(y)||\omega(y)| \|\omega\|_{L^\infty(B_1(y))} + |v(y)||\omega(y)| \|\omega\|_{L^q} \left(\int_{\{|x-y| > 1\}} \frac{dx}{|x-y|^{\frac{2q}{q-1}}} \right)^{\frac{q-1}{q}} \\
&\lesssim |v(y)||\omega(y)| \|\omega\|_{L^\infty(B_1(y))} + |v(y)||\omega(y)| \|\omega\|_{L^q} \left(r^{-\frac{2}{q-1}} dr \right)^{\frac{q-1}{q}} < +\infty \tag{2.41}
\end{aligned}$$

where we used the fact that $-\frac{2}{q-1} < -1$ if $\frac{3}{2} \leq q < 3$. From (2.35) we immediately obtain

$$\begin{aligned}
\int_{\mathbb{R}^3} \Phi(x, y) dy &= \omega(x) \cdot \left(\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{(x-y) \times (v(y) \times \omega(y))}{|x-y|^3} dy \right) \\
&= |\omega(x)|^2 \geq 0, \quad \forall x \in \mathbb{R}^3 \tag{2.42}
\end{aligned}$$

and combining this with (2.39), we deduce (2.37). On the other hand, using (2.34), we find

$$\begin{aligned}
\int_{\mathbb{R}^3} \Phi(x, y) dx &= \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(x) \cdot (x-y) \times (v(y) \times \omega(y))}{|x-y|^3} dx \\
&= \left(\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\omega(x) \times (x-y)}{|x-y|^3} dx \right) \cdot v(y) \times \omega(y) \\
&= v(y) \cdot v(y) \times \omega(y) = 0 \tag{2.43}
\end{aligned}$$

for all $y \in \mathbb{R}^3$, and combining this with (2.41), we have proved (2.38). This completes the proof of the claim.

If (1.9) holds, then from (2.42) and (2.43) provide us with

$$\int_{\mathbb{R}^3} |\omega(x)|^2 dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(x, y) dy dx = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \Phi(x, y) dx dy = 0.$$

Hence,

$$\omega = 0 \quad \text{on} \quad \mathbb{R}^3. \quad (2.44)$$

We remark parenthetically that in deriving (2.44) it is not necessary to assume that $\int_{\mathbb{R}^3} |\omega(x)|^2 dx < +\infty$, and therefore we do not need to restrict ourselves to $\omega \in L^2(\mathbb{R}^3)$. Hence, from (2.34) and (2.44), we conclude $v = 0$ on \mathbb{R}^3 . \blacksquare

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